Pareto Efficient Auctions with Interest Rates

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Abstract

We consider auction settings in which agents have limited access to monetary resources but are able to make payments larger than their available resources by taking loans with a certain interest rate. This setting is a strict generalization of budget constrained utility functions (which corresponds to infinite interest rates). Our main result is an incentive compatible and Pareto-efficient auction for a divisible multi-unit setting with 2 players who are able to borrow money with the same interest rate. The auction is an ascending price clock auction that bears some similarities to the clinching auction but at the same time is a considerable departure from this framework: allocated goods can be de-allocated in future and given to other agents and prices for previously allocated goods can be raised.

1 Introduction

One of the common simplifications in mechanism design is to assume that the monetary resources available to each agent are far more than the magnitude of the economic transaction for which a mechanism is being designed. This assumption is hidden in the *quasi-linear utility* model, which states that the utility for a certain outcome is its value minus the payment, no matter how large the payments are. While this assumption is fine for a variety of settings, this breaks if the magnitude of a transaction is large. For example, if one wants to buy a house or pay for college tuition, the amount of available funds may even be important than the actual value. A first order approximation to this issue is to consider the budget constrained utility function, which assumes that the utility is quasi-linear if the payment is below the budget and minus infinity otherwise. In the recent past, some progress has been made to develop our understanding of design of efficient mechanisms for budget-constrained utility functions. However, budget constrained utility functions fail to capture an important aspect of real life: to make large purchases one can *borrow* money. This is done by using financial instruments such as credit-cards, loans, mortgages, etc.

In this paper we investigate this gap – between the real-world scenario and our theoretical understanding of money-constrained mechanism design – by designing mechanisms for agents whose utility functions are non-linear convex functions of their payments. To be more specific, we consider utilities of the form $u = v(x) - \beta(\pi)$ where $v(x)$ is the value the agent derives from the implemented outcome, π is the payment and β is a non-linear convex function of the payment. This can be used to model scenarios with interest rates. For example, suppose that an agent has B monetary units readily available and has access to a bank that can lend him money with interest rate, say, $\gamma - 1$. This setting can be modeled by defining $\beta(\pi) := \pi$ for $\pi \leq B$ and $\beta(\pi) := B + \gamma(\pi - B)$ for $\pi > B$. To see this, if the agent is charged $\pi > B$ by the auctioneer, he can use B from his own funds and borrow $\pi - B$ from the bank. Since the agent has to pay $(\gamma - 1)(\pi - B)$ in interest to the bank, his final true cost will be $\beta(\pi) := B + \gamma(\pi - B)$.

The problem of designing auctions with non-quasi-linear utilities is notoriously hard, which explains why the vast majority of work in mechanism design assumes quasi-linearity. The progress in non quasi-linear utilities has been limited, with a few exceptions, mostly to budget-constraints and risk-aversion. In this paper, we seek to make progress in the problem of utilities that take interest rates into account. We consider the simplest possible setting: an auction for one unit of a divisible good between *two* agents who have access to B_i monetary units and can borrow at the same common interest rate.

Our main result is an individually-rational, incentive-compatible and Pareto-efficient auction for the setting described above. We note that the usual notion of efficiency in mechanism design - social welfare maximization - is impossible to be achieved in an incentive compatible manner in this setting (even if one allows for approximations). The reason is that a special case of our setting is that of budget constrained utility functions (by taking $\gamma \to \infty$), for which incentive compatible social welfare maximization is known to be impossible [DLN08]. In such cases, Pareto-efficiency becomes the natural way to achieve efficiency. Note for example that when B_i are very large or the interest rates tend to zero, Pareto-efficiency boils down to social-welfare maximization.

Since our setting is a strict generalization of budget constrained utilities, our auction must naturally be the clinching auction in the limit when the parameter γ tends to ∞, since the clinching auction is the unique Pareto-efficient incentive-compatible auction for budget constrained utilities. Our auction, therefore, contains many elements in common with Ausubel's clinching auction [Aus97], but at the same time is a considerable departure from the traditional clinching framework. The major similarities are that our auction can also be cast as an ascending price clock auction, which keeps at each step provisional allocation and payments and computes demands to determine what amount is safe to be given to each agent without violating demands of the others. Unlike the traditional clinching framework, however, the amount allocated to each agent is *not monotone* in the price, i.e., units can be allocated at some price and later taken away from an agent and given to the other. Secondly, the auction can, at a given price, increase the price charged for goods allocated at lower prices. Thirdly, the outcome is not determined when the price clock reaches the second-highest valuation. In our auction, the clock ascends all the way to the highest valuation and non-trivial changes in the allocation can happen all the way to the end.

Roadmap We design our ascending price clock auction in four steps:

1. In the first step, we notice that since the utility of the agents is not quasi-linear, Myerson's characterization of incentive compatible mechanisms doesn't apply. We get around this fact, by showing that the problem of designing auctions for players with utilities $u^\beta = v_i(x_i) - \beta_i(\pi_i)$ is equivalent to designing auctions for quasi-linear players in which the auctioneer is subject to taxes on his revenue. Given this transformation, we are able to use Myerson's characterization but now we need to be careful with the influence of the revenue taxes on the definition of Pareto-efficiency. This step corresponds to Lemma 2.1.

2. Next we provide a characterization of Pareto-efficient outcomes for settings in which the auctioneer's revenue is subject to taxes, by showing that Pareto-efficiency is equivalent to a certain no-trade condition. This step is standard in the design of Pareto-efficient auctions and corresponds to Lemma 2.3.

3. In Section 3 we cast the auction design problem as a differential equations problem and show it has a solution by using the Existence Theorem of First-Order Ordinary Differential Equations. We map back the solution to the differential equation to an auction satisfying all the desirable properties, which we call the *Taxed Differential Auction*. The auction is correct but has a somewhat cryptic description that lacks economic intuition.

4. This last issue is addressed in Section 4, in which we show that the auction in Section 3 can be interpreted as an ascending price clock auction. We re-define the auction as the *Taxed Ascending Auction*. This interpretation allows for an alternative proof of incentive compatibility and Pareto-efficiency of the auction. Unlike the proof in Section 3, the alternative proof provides a clean economic intuition of why such properties hold.

One might wonder why the third step is necessary given that the ascending price clock auction provides a self-contained proof of incentive-compatibility and Pareto-efficiency. While the third step is not mathematically necessary, it provides the necessary guidance for defining the ascending procedure in the fourth step. The authors feel that without the analysis in the third step, the auction definition in the fourth step would appear made out of thin air.

Note on the model and our contribution The model studied in this paper is admittedly simple: only 2 players and symmetric interest rates. We believe that the major contribution of this work is not a generally applicable framework but a proof of concept that progress can be made for sophisticated utility functions. From the technical perspective our contribution is two-fold: (a) we showcase the method of casting auction design as a differential equations problem and how to obtain an intuitive (from an economic interpretation perspective) auction out of the differential equation solution; (b) we produce a novel and non-trivial ascending auction with unique features. We believe the construction of our ascending auction is interesting in its own right.

Related Work In Bayesian settings, Maskin and Riley [MR84] were the pioneers in studying the design of mechanism with non-quasi-linear utilities. Their approach is also uses differential equations as the main mathematical tool, but since their setting is Bayesian (while ours is worst-case), both the equations and mechanisms obtained are quite different. Note that their mechanisms are depend heavily on the distribution from which types are drawn, where our mechanisms have no such dependency.

Also central to the study of the impact of financial constraints in auctions is the work of Che and Gale [CG98, CG06]. The authors study the impact of financial constraints in the Bayes-Nash equilibria of standard auctions (such as first and second price auctions). Besides begin Bayesian, another major difference to our work is the approach: while Che and Gale compare different auction formats, we take the mechanism design approach. An interesting connection between our papers is that both their work and ours provide a reduction from generic utilities to the quasi-linear setting. Yet, the reductions are different in nature and serve different purposes in the analysis. Che and Gale reduce to the quasi-linear setting by adding a fictitious risk-neutral bidder, while our reduction works by transferring the risk from the buyers to the auctioneer. Also, while our reduction is used to design dominant strategy incentive compatible mechanisms, Che and Gale use their reduction to establish revenue (Bayes-Nash-type) equivalence theorems.

The problem of mechanism design in which players have non-quasi-linear utility functions is well studied for the case in which the available goods are indivisible and each agent wants to acquire at most one good (unit-demand agents). For unit-demand agents, the stable marriage model of Gale and Shapley [GS62] together with the various flavors of the deferred-acceptance algorithm allow the design of efficient mechanisms with good incentive properties for a variety of settings with sophisticated utility functions. This is the route taken in the papers of Aggarwal, Muthukrishnan, Pal and Pal [AMPP09], Alaei, Jain and Malekian [AJM10], Morimoto and Serizawa [MS12]. Closer to our work is the paper by Dutting, Henzinger and Weber [DHW11], extend [AMPP09] the framework to account for hard and soft budget constraints, but stay within the realm of unit-demand bidders and heavily rely on stable matching constructions.

Our paper differs from this line of work in the sense that we look at the simplest setting in which the stable matching machinery is not available: divisible multi-unit auctions. Another major difference is that our goal is to design Pareto-efficient auctions while the previously mentioned papers focus on implementing envy-free outcomes. In the last sense, our paper is closer to the line of work initiated by Dobzinski, Lavi and Nisan [DLN08] and inspired by Ausubel's framework [Aus97]. In [DLN08] the authors design an incentive compatible, individually rational and Pareto-efficient auction for budget constrained utility functions. Their auction has been extended in multiple directions: [BCMX10] show how to elicit budgets truthfully, [FLSS11, CBHLS12] generalize the clinching to matching markets and [GMP12] to general polymatroidal environments and [GMP13] shows that the clinching auction allows for an online implementation. Closely related to this work is [GML14] in which we discuss how to design Pareto-efficient auctions where agents have *constrained quasi-linear utilities*, which mean that $u_i = v_i(x_i) - \pi_i$ when (x_i, π_i) belong to a certain admissible set $\mathcal{A}_i$ and $-\infty$ otherwise. This allows the authors to generalize the Polyhedral Clinching Auction to settings with average budgets and in general to settings in which the available budget is a function of the allocation. However, the model in [GML14] is not expressible enough to capture interest rates and other types of non-linearities.

Open Problems This paper leave open the question of whether it is possible to design incentive-compatible and Pareto-efficient auctions for the following cases: (i) 3 or more agents with symmetric interest rates; (ii) 2 players with agent-specific interest rates; (iii) 2 players with generic non-linear β_i-functions. The main issue with (i) and (ii) is that the characterization of Pareto-efficiency is less crisp in those cases. Also, most natural generalizations of the Taxed Ascending Auction in Section 4 break either incentive-compatibility or Pareto-efficiency when we move to 3 players or asymmetric interest rates.

2 Multi-Unit Auctions with Interest Rates

We say that an agent has β_i-utility if his utility for an outcome in which he is assigned a bundle x_i and is charged payment π_i is

$$u_i^\beta(x_i, \pi_i) = v_i(x_i) - \beta_i(\pi_i)$$

where $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex, strictly monotone function such that $\beta_i(\pi_i) \geq \pi_i$ for all $\pi_i \geq 0$. Intuitively, this measures his cost for acquiring π_i dollars. For quasi-linear players, β_i is simply the identity $\beta_i(\pi_i) = \pi_i$. For traditional hard budget constraints $\beta_i(\pi_i) = \pi_i$ for $\pi_i \leq B_i$ and $\beta_i(\pi_i) = \infty$ otherwise. We will denote its left and right derivatives at π_i as: $\beta_i'(\pi_i-)$ and $\beta_i'(\pi_i+)$.

We will be particularly interested in the case player i has B_i monetary units available to him and can borrow additional resources at an interest rate $r = \gamma_i - 1$. This is represented by:

$$\beta_i(\pi_i) = \begin{cases} \pi_i, & \pi_i \leq B_i \\ B_i + \gamma_i(\pi_i - B_i), & \pi_i \geq B_i \end{cases}$$

which is depicted in Figure 1.

We consider this problem in the context of multi-unit auctions with divisible goods: the allocation of each player is a real number $x_i \in [0, 1]$ and $v_i(x_i) = v_i \cdot x_i$. The set of feasible allocations is given by $F = \{x \in \mathbb{R}^n_+; \sum_i x_i \leq 1\}$.

We assume that the β-functions are public information and that values are private. So, fixed β_i for each player, a mechanism for multi-unit auctions is a pair of mappings $x : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $\pi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Incentive compatibility and individual rationally have their usual meaning: each player maximizes his utility by reporting his true value and upon reporting his true value, he has non-negative utility. Since this is *not* a quasi-linear setting, Myerson's characterization [Mye81] doesn't directly apply.

We say that an allocation (x, π) is *Pareto-efficient* if there is no alternative allocation (x', π') where each agents utility is at least as good as in the original allocation, the revenue of the auctioneer

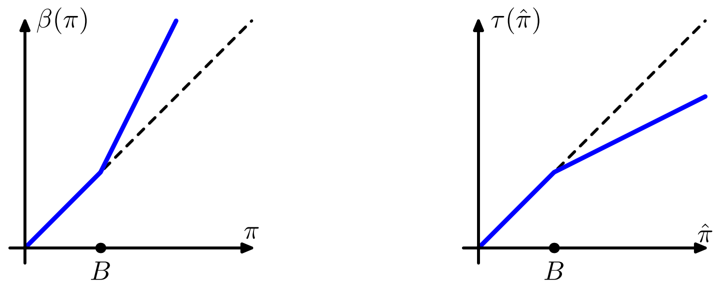

Figure 1: Interest rates represented by a β-function and its corresponding τ-taxation for $B = 1$ and $\gamma = 2$.

is at least as good and at least one of them strictly improves. Formally: there is no alternative allocation, where:

$$v_i \cdot x_i' - \beta_i(\pi_i') \geq v_i \cdot x_i' - \beta_i(\pi_i'), \forall i, \quad \sum_i \pi_i' \geq \sum_i \pi_i, \quad \sum_i u_i^\beta(x_i', \pi_i') + \pi_i' > \sum_i u_i^\beta(x_i, \pi_i) + \pi_i$$

Before designing an auction for this setting, it is instructive to see why simple designs fail to achieve the desirable properties. A natural auction for this setting is the one holds sequential second price auctions for infinitesimal parts of the good: for each infinitesimal part of the good, allocate to the buyer with largest marginal valuation for that piece, where the marginal valuation for buyer i is v_i if he hasn't reached his budget and v_i/γ_i otherwise. This design, while simple, doesn't yield an incentive compatible auction. Consider the example with 1 unit of a divisible good and 2 players with values: $v_1 = 1 + \epsilon$, $v_2 = 1$, $\gamma_1 = \gamma_2 = 1/\epsilon$ and $B_1 = B_2 = 1/3$. In the simple design, for the first 1/3 of the good, both agents bid 1, so agent 1 acquires the first 1/3 and pays his entire budget and from this point on, he bids $v_1/\gamma_1 = O(\epsilon)$, so agent 2 acquires the remaining 2/3 of the good. Now, if agent 1 were to shade his bid to $1-\epsilon$, the situation would be reversed and agent 2 would be allocated the first $1/3 + O(epsilon)$ of the good and agent 1 would be alllocated 2/3. Notice that the mechanism can't even be made truthful by changing the payment rule, since the allocation is not monotone.

Indeed, there is a deeper reason why it is not possible to get a simple Pareto-efficient auction in this setting. Once $\gamma \to 0$, any auction that is Pareto-efficient must recover the Clinching Auction in [DLN08], since it is the unique incentive compatible auction with hard budget constraints. We refer to [DL14] for a in-depth discussion of why simple designs typically fail to achieve Pareto-efficiency in an incentive-compatible manner.

2.1 Equivalence between β_i-utilities and quasi-linear settings with taxation

First we show that designing a Pareto-efficient auction for agents with β_i-utilities is equivalent to designing Pareto-efficient auctions for quasi-linear settings in which the auctioneer is subject to taxation over his revenue. The central idea behind the equivalence is to consider *effective payments* $\hat{\pi}_i = \beta_i(\pi_i)$ and design an auction in terms of effective payments charged to the buyer. The issue with that is that $\hat{\pi}_i$ effective payment results in revenue $\pi_i = \beta_i^{-1}(\hat{\pi}_i)$ paid to the auctioneer. Below, we formally define a setting with taxation:

Quasi-linear settings with taxation Consider the problem where there are n agents with quasi-linear utilities $u_i(x_i, \pi_i) = v_i \cdot x_i - \pi_i$ and for each agent there is a taxation function $\tau_i : \mathbb{R}_+ \to \mathbb{R}_+$ that is strictly monotone, concave and has $\tau_i(\pi_i) \leq \pi_i$ for all $\pi_i \geq 0$.

An auctioneer is taxed on the income received from each buyer according to the τ_i-taxation functions. His revenue is therefore $\sum_i \tau_i(\pi_i)$. An allocation for this setting is Pareto-efficient if there is no alternative allocation (x', π') such that:

$$v_i \cdot x_i' - \pi_i' \geq v_i \cdot x_i' - \pi_i', \forall i, \quad \sum_i \tau_i(\pi_i') \geq \sum_i \tau_i(\pi_i), \quad \sum_i u_i(x_i', \pi_i') + \tau_i(\pi_i') > \sum_i u_i(x_i, \pi_i) + \tau_i(\pi_i)$$

Now, given a mechanism (x, π) for the setting with β-utilities, consider the mechanism $(x, \hat{\pi})$ for the setting with quasi-linear utilities and τ-taxes with $\tau = \beta^{-1}$ in which $\hat{\pi}_i(v) = \beta_i(\pi_i(v))$. It follows directly from the definitions that (x, π) is incentive-compatible, individually-rational and Pareto optimal for the β-utilities setting iff $(x, \hat{\pi})$ is such for the τ-taxes setting.

Lemma 2.1 (equivalence)**.** *Given β_i strictly monotone convex functions and $\tau_i = \beta_i^{-1}$ strictly monotone concave functions, there is an incentive-compatible, individually-rational and Pareto-optimal mechanism for agents with β-utilities iff there is a mechanism with the same properties for quasi-linear agents where the auctioneer pays τ-taxes on his revenue.*

The main advantage of Lemma 2.1 is that it allows us to use Myerson's characterization of incentive compatibility, since agents are quasi-linear. We recall Myerson's characterization:

Lemma 2.2 ([Mye81])**.** *A single-parameter mechanism for quasi-linear agents defined by $x : \mathbb{R}_+^n \to [0,1]^n$ and $\pi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is individually-rational and incentive compatible iff:*

- *$x_i(v_i, v_{-i})$ is monotone non-decreasing in v_i for any fixed v_{-i};*
- *payments are such that $\pi_i(v_i, v_{-i}) = v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(u, v_{-i}) du$.*

2.2 Characterizing Pareto-efficient outcomes

Our first step is to provide a characterization of Pareto-efficient outcomes for settings with taxation. The following Lemma is a version of Proposition 2.4 in [DLN08] for the utility model studied in this paper. Versions of this lemma for different settings have been provided in [BCMX10, GMP12, GML14].

We recall the reader that $\tau_i'(\pi_i-)$ and $\tau_i'(\pi_i+)$ denote the left and right derivatives of the taxation function τ_i.

Lemma 2.3 (Pareto-optimality characterization)**.** *Consider n agents with quasi-linear utilities and an auctioneer that obtains revenue $\sum_i \tau_i(\pi_i)$ from the outcome (x, π) and a divisible multi-unit auctions setting, i.e. $\sum_i x_i(v) \leq 1$. Such outcome is Pareto-optimal iff (i) all goods are sold, i.e., $\sum_i x_i(v) = 1$ and (ii) no trade is possible, i.e., for all agents $i \neq j$ such that $x_i > 0$, it holds that $v_i \cdot \tau_i'(\pi_i-) \geq v_j \cdot \tau_j'(\pi_j+)$.*

Proof. For the $(\Rightarrow)$ direction, if the allocation doesn't sum to one, one can allocate the left-over goods to some player for free, improving his utility. Also, if for some pair $x_i > 0$ and $v_i \cdot \tau_i'(\pi_i-) < v_j \cdot \tau_j'(\pi_j+)$, then there is some ϵ for which $\tau_i(\pi) - \tau_i(\pi_i - v_i\epsilon) < \tau_j(\pi_j + v_j\epsilon) - \tau_j(\pi_j)$. So, consider the outcome (x', π') where for $k \neq i, j$, $x_k' = x_k$ and $\pi_k' = \pi_k$, $x_i' = x_i - \epsilon$, $\pi_i' = \pi_i - v_i\epsilon$, $x_j' = x_j + \epsilon$ and $\pi_j' = \pi_j + v_j\epsilon$. All utilities are still the same, and the auctioneer revenue improved.

For the $(\Leftarrow)$ direction, consider an allocation (x, π) with the two characterization properties and let (x', π') be a Pareto-improvement. If some players utility strictly improved we can always increase his payment so that the utilities are the same as before for all agents but the revenue of the auctioneer improved. Assuming that, we now compare the revenue in both allocations.

$$0 \geq \sum_i \tau_i(\pi_i) - \tau_i(\pi_i') \geq \sum_{i;\pi_i > \pi_i'} \tau_i'(\pi_i-) \cdot (\pi_i - \pi_i') + \sum_{i;\pi_i < \pi_i'} \tau_i'(\pi_i+) \cdot (\pi_i - \pi_i') =$$

$$= \sum_{i;\pi_i > \pi_i'} \tau_i'(\pi_i-) \cdot v_i \cdot (x_i - x_i') + \sum_{i;\pi_i < \pi_i'} \tau_i'(\pi_i+) \cdot v_i \cdot (x_i - x_i') \geq 0$$

Since: $\sum_{i;\pi_i>\pi'_i} x_i - x'_i = \sum_{i;\pi'_i>\pi_i} x'_i - x_i$ and the coefficients $\tau'_i(\pi_i-) \cdot v_i$ are larger then the coefficients $\tau'_i(\pi_i+) \cdot v_i$ by the characterization condition. So, this implies that $\sum_i \tau_i(\pi_i) - \tau_i(\pi'_i) = 0$ contradicting the fact that (x', π') is a Pareto improvement. ■

3 Existence of 2-agents Multi Unit Auctions with τ-Taxation

First we consider the problem of designing an auction for $n = 2$ agents with the same taxation function:

$$\tau_1(\pi) = \tau_2(\pi) = \begin{cases} \pi, & \pi \leq B \\ B + \gamma^{-1} \cdot (\pi - B), & \pi \geq B \end{cases}$$

for some constant $\gamma > 1$. The reader should note that by Lemma 2.1 this is equivalent to the β-utilities described in Section 2.

Let's assume we have an incentive compatible, individually rational and Pareto optimal auction $x(v_1, v_2), \pi(v_1, v_2)$. We will deduce its exact from by applying the characterization of Pareto-efficient outcomes in Lemma 2.3 together with Myerson's characterization in Lemma 2.2. We proceed in a sequence of steps, captured by a sequence of facts:

Fact 3.1. *In the region $v_2 < \min\{B, v_1\}$, $x(v_1, v_2) = (1, 0)$. Analogously, in the region $v_1 < \min\{B, v_2\}$, $x(v_1, v_2) = (0, 1)$.*

Proof. In the region $v_1 < B, v_2 < B$ the payments are all also $\pi_1 < B$ and $\pi_2 < B$, therefore, condition (ii) in Lemma 2.3 becomes $x_i > 0 \Rightarrow v_i \geq v_j$. Since $B > v_1 > v_2$, then: $x(v_1, v_2) = (1, 0)$. By monotonicity (first condition in Lemma 2.2) for any $\tilde{v}_1 \geq v_1$ we have $x(\tilde{v_1}, v_2) = (1, 0)$. ■

Fact 3.2. *In the region $v_2 < \gamma^{-1} v_1$, $x(v_1, v_2) = (1, 0)$. Analogously, in the region $v_1 < \gamma^{-1} v_2$, $x(v_1, v_2) = (0, 1)$.*

Proof. Since the left and right derivatives of τ are either 1 or γ^{-1} we have that if $x_2 > 0$, then: $v_2 \geq v_2 \cdot \tau'(\pi_2-) \geq v_1 \cdot \tau'(\pi_1+) \geq \gamma^{-1} \cdot v_1$. So for $v_2 < \gamma^{-1} v_1$, the only allocation that is Pareto optimal is $(1, 0)$. ■

Fact 3.3. *In the region $\{v; v_1 > v_2 > \gamma^{-1} v_1; v_2 > B\}$ the allocation $x(v) \neq (0, 1)$. Analogously, in the region $\{v; v_2 > v_1 > \gamma^{-1} v_2; v_1 > B\}$ the allocation $x(v) \neq (1, 0)$.*

Proof. Assume we have $v = (v_1, v_2)$ with $v_1 > v_2 > \gamma^{-1} v_1$ and $v_2 > B$ and $x(v) = (0, 1)$. Then by individual rationality, $\pi_1(v) = 0$, so the Pareto-optimality condition states that $v_2 \geq v_2 \cdot \tau'(\pi_2) \geq v_1 \cdot \tau'(0) = v_1$, which contradicts that $v_1 > v_2$. ■

Fact 3.4. *In the region $\{v; v_1 > v_2 > \gamma^{-1} v_1; v_2 > B\}$ the allocation $x(v_1, v_2) \neq (1, 0)$. Analogously, in the region $\{v; v_2 > v_1 > \gamma^{-1} v_2; v_1 > B\}$ the allocation $x(v) \neq (0, 1)$.*

Proof. Assume we have $v = (v_1, v_2)$ with $v_1 > v_2 > \gamma^{-1} v_1$ and $v_2 > B$ and $x(v_1, v_2) = (1, 0)$. By Myerson's integral, $\pi_1(v_1, v_2) = \int_0^{v_1} x_1(v_1, v_2) - x_1(u, v_2) du \geq \int_0^B x_1(v_1, v_2) - x_1(u, v_2) du = B$. If $\pi_1(v) > B$, then since $\pi_2(v) = 0$ by individually rationally, the Pareto conditions would imply that $v_1 \gamma^{-1} \geq v_2$, which contradicts that $v_2 > \gamma^{-1} v_1$. Now, if $\pi_1(v) = B$, then the only way the Myerson integral can be B is that $x(u, v_2) = (1, 0)$ for all $B < u \leq v_2$. But this would imply a $(1, 0)$ allocation in the region $\{v; v_2 > v_1 > \gamma^{-1} v_2; v_1 > B\}$, contradicting the previous fact. ■

Fact 3.5. *In the region $\{v; v_1 > v_2 > \gamma^{-1} v_1; v_2 > B\}$ either $\pi_1(v) = B$ and $\pi_2(v) \leq B$ or $\pi_1(v) \geq B$ and $\pi_2(v) = B$. Analogously, in the region $\{v; v_2 >_1 > \gamma^{-1} v_2; v_1 > B\}$ either $\pi_2(v) = B$ and $\pi_1(v) \leq B$ or $\pi_2(v) \geq B$ and $\pi_1(v) = B$.*

Proof. By the previous two facts, every allocation $x(v)$ for v in the region $\{v; v_1 > v_2 > \gamma^{-1}v_1; v_2 > B\}$ has both $x_1(v) > 0$ and $x_2(v) > 0$. So by Pareto-optimality conditions, it must be the case that: $v_1 \cdot \tau'(\pi_1-) \geq v_2 \cdot \tau'(\pi_2+)$ and $v_2 \cdot \tau'(\pi_2-) \geq v_1 \cdot \tau'(\pi_1+)$. Consider the following cases:

- If $\pi_1 < B$, then $v_1 = v_1 \cdot \tau'(v_1+) \leq v_2\tau'(v_2-) \leq v_2$, which contradicts that $v_1 > v_2$.
- If $\pi_2 > B$, then $v_2\gamma^{-1} = v_2\tau'(v_2-) \geq v_1 \cdot \tau'(v_1+) \geq v_1 \cdot \gamma^{-1}$, also contradicting that $v_1 > v_2$.
- If $\pi_1 > B$ and $\pi_2 < B$, then the two conditions imply that $v_1\gamma^{-1} = v_2$, which contradicts that $v_2 > \gamma^{-1}v_1$.

■

Let's summarize what we proved so far. In the regions $\{v; \min v_i \leq B\}$, $\{v; v_2 \geq \gamma v_1\}$ and $\{v; v_1 \geq \gamma v_2\}$ we already pinned down the allocation as $(1,0)$ whenever $v_1 \geq v_2$ and $(0,1)$ otherwise (Facts 3.1 and 3.2). This corresponds to regions A and A' in Figure 3. For the remaining regions we proved that the allocation and payments must be such that if $v_i > v_j$ then either $\pi_i = B \geq \pi_j$ or $\pi_i \geq B = \pi_j$ (Fact 3.5). From Fact 3.5 it also follows that ensuring this property is enough to get Pareto-optimality.

Next we formulate the problem of satisfying the properties outlined above in the remaining region $R = \{v \in \mathbb{R}^2_+; \min(v_1, v_2) \geq B; v_2 \geq \gamma^{-1}v_1; v_1 \geq \gamma^{-1}v_2\}$ as a differential equation. In order to give an intuition to the reader about the solution, we first solve the simpler problem of designing an auction satisfying the desirable properties in the smaller region $R' = [B, B+\epsilon]^2$ for some small ϵ.

Given a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ we will denote by $\partial_i f : \mathbb{R}^2 \to \mathbb{R}$ the derivative with respect to the i-th component. Given an univariate differentiable function $g : \mathbb{R} \to \mathbb{R}$, we will denote its derivative by g'. We denote left and right limits by $g(x-) = \lim_{\delta\uparrow 0} g(x+\delta)$ and $g(x+) = \lim_{\delta\downarrow 0} g(x+\delta)$.

Motivation: We will enforce in region $R' = [B, B+\epsilon]^2$ the following properties:

- (i) for $v_1 > v_2$, $\pi_1(v) = B$, $\pi_2(v) \leq B$
- (ii) for $v_2 > v_1$, $\pi_1(v) \leq B$, $\pi_2(v) = B$
- (iii) the allocation should be symmetric, i.e., $x_1(v_1, v_2) = x_2(v_2, v_1)$.

The properties (i) and (ii) are sufficient for Pareto-efficiency by Fact 3.5. We choose to enforce them instead of other combinations since it leads to a more natural differential equation to solve, as we will see soon. We also choose to enforce (iii) since it makes the problem more tractable. A different reason for choosing (i), (ii) and (iii) is that those are the conditions satisfied by the traditional clinching auction (see [DLN08, BCMX10, GMP13]). Given that clinching auction should be the limit of this auction as $\gamma \to \infty$, this is a natural first choice.

Now that we assume that properties (i), (ii) and (iii) hold, we note that since $\pi_1(v_1, v_2) = B$ in $R' \cap \{v; v_1 > v_2\}$ we have $\partial_1 \pi_1(v_1, v_2) = 0$ and therefore $\partial_1 x_1(v_1, v_2) = 0$. So we can write $x(v_1, v_2) = (1 - \hat{\chi}(v_2), \hat{\chi}(v_2))$ in that region. By (iii), we can write $x(v_1, v_2) = (\hat{\chi}(v_1), 1 - \hat{\chi}(v_1))$ on $R' \cap \{v; v_2 > v_1\}$. Now, we can use Myerson's Lemma to write the constraint that $\pi_1(v_1, v_2) = B$ in $R' \cap \{v; v_1 > v_2\}$ in integral form:

$$B = \pi_1(v_1, v_2) = v_1 x_1(v_1, v_2) - \int_0^{v_1} x_1(u, v_2)du = v_1(1 - \hat{\chi}(v_1)) - \int_B^{v_1} \hat{\chi}(w)dw$$

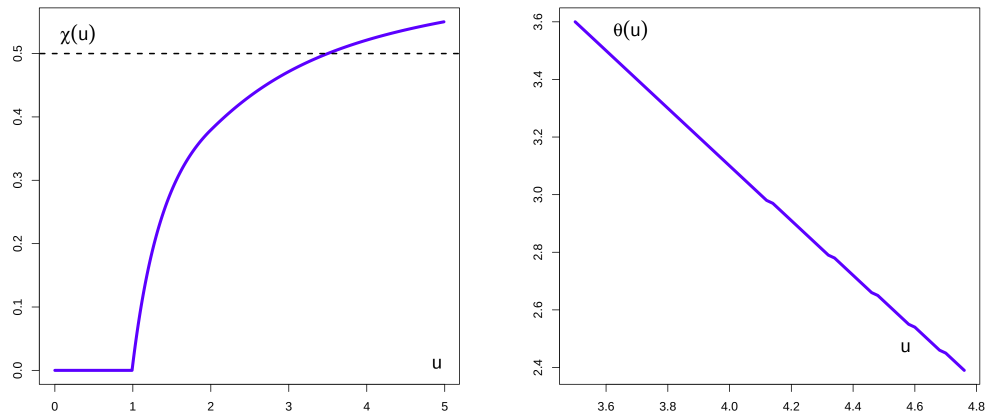

Figure 2: Numerical solution of $\chi(u)$ and $\theta(u)$ for $B = 1$ and $\gamma = 2$.

A function satisfying this expression can be obtained by derivating the above expression: $\hat{\chi}'(u) = \frac{1}{u}[1 - 2\hat{\chi}(u)]$ and solving the resulting differential equation with boundary condition $\hat{\chi}(B) = 0$. A solution to this differential equation exists and is unique by the Existence Theorem of First Order Differential Equations (see [Per01, Cod12]).

Now, we can check that x defined this way paired with π obtained from the Myerson integral, results in a incentive compatible, individually rational and Pareto-efficient mechanism on R'. In order to check that it is incentive compatible and individually rational, we only need to check that x is monotone. For $v_1 > v_2$, $\partial_2 x_2 = \hat{\chi}'$ which is positive whenever $1 - 2\hat{\chi} > 0$ which holds for small ϵ since $\hat{\chi}$ is continuous. Also, for $v_1 < v_2$, $\partial_2 x_2 = 0$ and fixed v_1, $x_2(v_1, v_1-) = \hat{\chi}(v_1) < 1 - \hat{\chi}(v_1) = x_2(v_1, v_1+)$ whenever $\hat{\chi} \leq 1/2$.

In order to check Pareto-efficiency, we only need to check conditions (i) and (ii) above. Since they are symmetric, it is enough to check (i). The condition $\pi_1(v) = B$ holds by design. For $\pi_2(v) \leq B$, notice that: $\pi_2(v) \leq \pi_2(v_1, v_1) = B$, where the first inequality comes from monotonicity and the second by the definition of χ.

Full analytic construction: Now we show how to extend x to the entire region R. The first challenge is that R is not a square like R'. The second challenge is that we don't rely on ϵ being small to claim that $\hat{\chi} \leq 1/2$. Below, we show how to address both issues.

First, we extend $\hat{\chi}$ appropriately by defining the following Ordinary Differential Equations Problem. As usual, a solution exists and is unique by the Existence and Uniqueness Theorem for ODE. Define $\chi : \mathbb{R}_+ \to \mathbb{R}$ to be the unique function satisfying the following properties:

$$\chi(u) = 0 \text{ for } u \in [0, B] \quad \text{and} \quad u(1 - \chi(u)) - \int_{\gamma^{-1}u}^{u} \chi(w)dw = B \text{ for } u \geq B$$

where the last integral expression is equivalent to:

$$\chi'(u) = \frac{1}{u}\left[1 - 2 \cdot \chi(u) + \gamma^{-1} \cdot \chi(\gamma^{-1} \cdot u)\right]$$

In Figure 2 we solve it numerically for $B = 1$ and $\gamma = 2$.

Clearly χ is continuous and differentiable for $u > B$. Also, let $v' = \min\{u > B; \chi(u) \geq 1/2\}$, then it is clear that for $u \leq v'$, $\chi'(u) \geq 0$ and therefore the function is monotone. For $R' \cap \{\max(v_1, v_2) \leq v'\}$ we will implement the allocation ruled $x(v) = (1 - \chi(v_2), \chi(v_2))$ in $R \cap \{v; v_1 > v_2\}$ and $x(v) = (\chi(v_1), 1 - \chi(v_1))$. By the same arguments used in the motivation, the auction has all the desirable properties in this rage.

Outside this range we can't implement the same allocation even if χ is non-decreasing, since monotonicity also requires that $x(v_1, v_1-) \leq x(v_1, v_1+)$ which would be violated if $\chi > 1/2$.

In what follows we discuss how to get around this problem. First define v'' to be $\min\{u \geq B; \gamma^{-1}u \cdot \chi(\gamma^{-1}u) = B\}$.

Fact 3.6. *Given v' and v'' previously defined, $\gamma^{-1}v'' \leq v' \leq v''$.*

Proof. For $v' \leq v''$ notice that for $u \leq v'$ we have:

$$\begin{aligned} B = u(1-\chi(u)) - \int_{\gamma^{-1}u}^{u} \chi(w)dw \geq u\chi(u) - \int_{\gamma^{-1}u}^{u} \chi(w)dw \\ = \gamma^{-1}u\chi(u) + \int_{\gamma^{-1}u}^{u} (\chi(u) - \chi(w))dw > \gamma^{-1}u \cdot \chi(\gamma^{-1}u) \end{aligned}$$

For $\gamma^{-1}v'' \leq v'$, first note that $v' \geq 2B$ since:

$$B = v'/2 - \int_{\gamma^{-1}v'}^{v'} \chi(w)dw \leq v'/2$$

Therefore for $u = \gamma v'$, we have: $\gamma^{-1}u \cdot \chi(\gamma^{-1}u) = v' \cdot \chi(v') = v'/2 \geq B$. ∎

Now, define the function $\theta : [v', v''] \to \mathbb{R}$ such that:

$$\theta(u) = \min\{t;\ t \cdot \chi(t) - \int_{\gamma^{-1}u}^{t} \chi(w)dw \geq B\}$$

For $B = 1$ and $\gamma = 2$ example in Figure 2, $v' = 3.5$ and $v'' = 4.76$. The θ function is also depicted in Figure 2. First we consider two of its properties:

Fact 3.7. *For the function θ defined above, $\theta(v') = v'$, $\theta(v'') = \gamma^{-1}v''$ and θ is non-increasing continuous and differentiable function in the range $[v', v'']$.*

Proof. The fact that $\theta(v') = v'$ and $\theta(v'') = \gamma^{-1}v''$ follow from the definitions of v' and v''. In order to see that θ is non-increasing, notice that:

$$H(u, t) = t \cdot \chi(t) - \int_{\gamma^{-1}u}^{t} \chi(w)dw$$

is increasing in u, since $\partial_u H(u, t) = \gamma^{-1} \cdot \chi(\gamma^{-1}u) > 0$. ∎

Now we are ready to present our full mechanism. We use B, γ, v' and v'' to define regions in the space of valuation functions and in each region we define how to allocate the goods using $\chi(\cdot)$ and $\gamma(\cdot)$. The regions are depicted in Figure 3.

Taxed differential auction

Setting: one unit of a divisible good and two agents with values v_1, v_2 per unit and τ-taxation such that $\tau(\pi) = \pi$ for $\pi \leq B$ and $\tau(\pi) = B + \gamma^{-1}(\pi - B)$.

Allocation rule: Assume wlog that $v_1 \geq v_2$ (if now, swap indices):

[Region A] If $v_2 \leq \max(B, \gamma^{-1}v_1)$, choose allocation $x(v) = (1, 0)$.

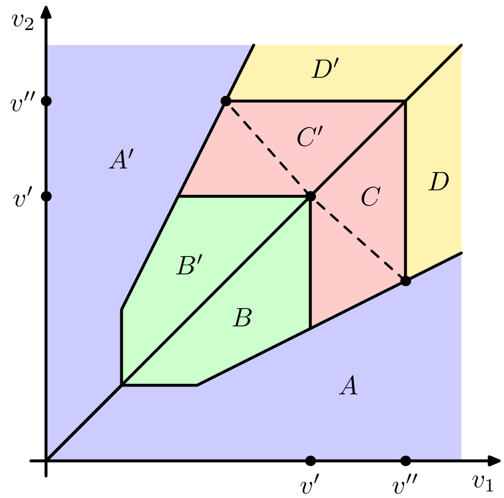

Figure 3: Regions for which we define the allocation rule of the mechanism. The dashed line crossing region C corresponds to the $\theta(\cdot)$ function.

[Region B] Else if $v_1 \leq v'$, allocate $x(v) = (1 - \chi(v_2), \chi(v_2))$.

[Region C] Else if $v_1 \leq v''$, allocate $x(v) = (1 - \chi(\min(v_2, \theta(v_1))), \chi(\min(v_2, \theta(v_1))))$.

[Region D] Else (if $v_1 \geq v''$), allocate $x(v) = (1 - \frac{B}{\gamma^{-1} v_1}, \frac{B}{\gamma^{-1} v_1})$.

Payments: Compute payments using Myerson's integral.

Theorem 3.8. *The mechanism presented above is an incentive compatible, individually rational and Pareto efficient mechanism for 2 players with τ-taxation functions where $\tau(\pi) = \pi$ for $\pi \leq B$ and $\tau(\pi) = B + \gamma^{-1}(\pi - B)$.*

Proof. First, in order to check incentive compatibility and individual rationality, it is enough to check that the allocation is monotone, since payments are computed by the Myerson integral. The allocation is clearly monotone within each region, what can be seen by simply inspecting the formulas. So, we only need to check if monotonicity holds on the boundary of the regions. Since x_1 and x_2 are symmetric, we only check monotonicity for x_2. We refer to the regions corresponding to A, B,C and D in the $v; v_2 > v_1$ region by A', B', C', D'.

- boundary between A and any other region: since the allocation in A is $(0, 1)$, if v_2 increases his allocation can't decrease and if v_1 decreases, his allocation can't increase. So, monotonicity holds.

- boundary between B and B': $x_2(v, v-) = \chi(v) \leq 1 - \chi(v) = x_2(v, v+)$ since by the definition of v', $\chi(v) \leq 1/2$.

- boundary between B and C (and boundary between B' and C'): the allocation is continuous.

- boundary between C and C': $x_2(v, v-) = \chi(\theta(v)) \leq 1 - \chi(\theta(v)) = x_2(v, v+)$. Where the inequality comes from the fact that $\theta(v) \leq \theta(v') = v'$ (since $\theta(\cdot)$ is non-increasing), so: $\chi(\theta(v)) \leq \chi(v') = 1/2$.

- boundary between C and D (and boundary between C' and D'): the fact that for v'', $\theta(v'') = \gamma^{-1}v''$ and that $\chi(v'') = B/(\gamma^{-1}v'')$ implies that the allocation is also continuous in that boundary.

- boundary between D and D': $x_2(v, v-) = \frac{B}{\gamma^{-1}v} \leq 1 - \frac{B}{\gamma^{-1}v} = x_2(v, v+)$ iff $v \geq 2 \cdot \frac{B}{\gamma^{-1}}$. This is true since $v \geq v'' \geq \frac{B}{\gamma^{-1}v''}$, where the second inequality comes from the fact that $\frac{B}{\gamma^{-1}v''} = \chi(v'') \leq \chi(v') = \frac{1}{2}$.

Finally, we show that the mechanism is Pareto-efficient by showing that the conditions in Fact 3.5 hold in each regions B, C, D.

- In region B and in region C below the dashed line, $x(v) = (1 - \chi(v_2), \chi(v_2)$. The payment of player 1 is given by the following Myerson integral:
$$\begin{aligned}\pi_1(v) &= \int_0^{v_1} [x_1(v_1, v_2) - x_1(u, v_2)]du = \int_0^{v_2} [x_1(v_2+, v_2) - x_1(v_1, u)]du \\ &= v_2(1 - \chi(v_2)) - \int_{\gamma^{-1}v_2}^{v_2} \chi(w)dw = B\end{aligned}$$
Now, for player 2, notice that:
$$\pi_2(v_1, v_2) \leq \pi_2(v_1, v_1+) = B$$
which satisfy the Pareto-efficiency conditions in Fact 3.5,

- In region C above the dashed line we have $\pi_1(v_1, v_2) \geq \pi_1(v_2+, v_2) = B$. For π_2 we have that $\pi_2(v_1, v_2) = \pi_2(v_1, \theta(v_1))$ since the allocation is constant for (v_1, u) and $u \in [\theta(v_1), v_1]$. Now:
$$\pi_2(v_1, \theta(v_1)) = \theta(v_1) \cdot \chi(\theta(v_1)) - \int_{\gamma^{-1}v_1}^{\theta(v_1)} \chi(w)dw = B$$
by the definition of θ. So again we satisfy the Pareto efficiency conditions in Fact 3.5.

- In region D, by applying the Myerson integral, we get $\pi_2(v_1, v_2) = \gamma^{-1}v_1 \cdot \frac{B}{\gamma^{-1}v_1} = B$. For player 1, $\pi_1(v) \geq \pi_1(v_2, v_2) = B$, which satisfies the conditions again.

■

4 Ascending Price Clock Auction

Theorem 3.8 constructs an incentive compatible, individually rational and Pareto-efficient auction for a setting with τ-taxation using a very laborious method. We went through the following steps: (i) identify the properties we want to satisfy and characterize them (Lemmas 2.1 and 2.3). (ii) divide the valuation space in regions and pin down the allocation for each regions in which this is straightforward (Facts 3.1 to 3.5. (iii) for the remaining region, cast the problem as a differential equation and use its solution to design the allocation rule.

As it is usually the case with explicit descriptions of auctions, the formulas are somewhat cryptic and offer little intuition that can be generalized to other settings. In this section, we interpret the previous section as an ascending price clock auction and generalize it to non-symmetric β-functions. The ascending auction bears some resemblance to Ausubel's clinching auction [Aus97] – but it has unique - features absent from traditional clinching designs. For example, *clinched* goods can be re-assigned later or have their price increased.

Before we begin the analysis, one might wonder why one should bother with solving differential equations when an ascending auction is available. We note to the reader that the design of the ascending auction has various small details and the whole auction fails to work unless they are

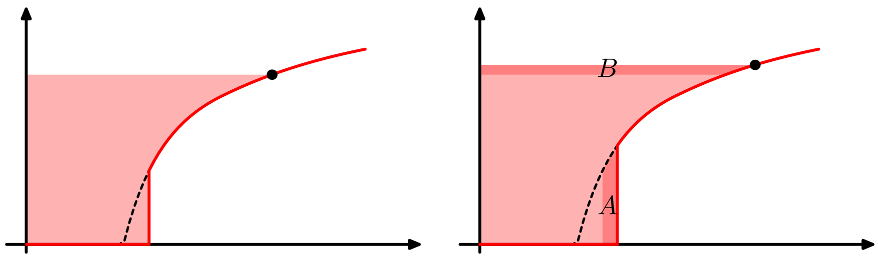

Figure 4: Area above the ψ-curve corresponding to payments $\pi_2(p, p-) = p\psi(p) - \int_{\gamma^{-1}p}^{p} \psi(u)du$ for $p = 2.5$ and $p = 2.8$..

all correctly adjusted. Getting them all right in first place is considerably hard and this is the point where solving a differential equation can guide our design. The second reason we do this is to showcase the approach of writing desirable properties as a differential equation and explicitly solving them. This approach is used in [GMP12] to prove an impossibility result. Here we show how to use the same approach to obtain a positive result.

Intuition: from a differential equation to a clinching auction The way an ascending price auction works is that a variable $p \in \mathbb{R}_+$ known as the price clock ascends and for each value of p we simulate the outcome of the auction in the valuation profile (p, p). Or more generally, we ask ourselves: "what part of the allocation can we fix is the only information available to us is that the valuations are at least p." We look at the auction designed in the previous section from this perspective: for each p, we look at allocation of the auction for price p and we notice that we have:

$$\psi(p) := x_1(p-, p) = x_2(p, p-) = \begin{cases} 0, & 0 \leq p \leq B \\ \chi(p), & B \leq p \leq v' \\ \chi(\theta(p)), & v' \leq p \leq v'' \\ B/(\gamma^{-1}p), & v'' \leq p \end{cases}$$

The first issue one can notice is that unlike in Ausubel's clinching auction $\psi(p)$ is not monotone. It is monotone, however, in the $[0, v']$ range. For that range, we would like to define payments as $\int_0^p p \cdot \partial\psi(p)dp = p \cdot \psi(p) - \int_0^p \psi(u)du$ as it is done in clinching auctions, where the payment is the sum (integral) of p times the amount received by the agent when the clock had that price. Unfortunately those don't correspond to the auction payments as well. Even for the $[B, v']$ range, $\pi_2(p, p-) = p\psi(p) - \int_{\gamma^{-1}p}^{p} \psi(u)du$. As usually done for the standard payment formula, the payments here can also be written as the area above the allocation curve[1], which we depict in Figure 4.

In Figure 4 we depict the payment for p and $p + \epsilon$ and we can see that the area representing the payments increases in two different ways: (i) the area marked as B corresponds to $p \cdot \partial\psi(p)dp$ which is the usual increase in price; (ii) the area marked as A represents an increase $\gamma^{-1}dp \cdot \chi(\gamma^{-1}p)$, which corresponds to increasing the price all units sold at price smaller or equal to $\gamma^{-1}p$.

We now use this insight to motivate the design of a *clinching-like* auction. For now, assume that all values are multiples of ϵ and that the price clock ascends also in ϵ-increments. The auction starts with $x_1(0) = x_2(0) = 0$, $\pi_1(0) = \pi_2(0) = 0$ and $p = 0$. We will keep at $x_i(p)$ and $\pi_i(p)$ the total allocation and payments of agent i between price zero and price p. We update as follows: after we fix $x_i(p)$ and $\pi_i(p)$, we (i) increase the price to $p+\epsilon$; (ii) increase the price of the goods purchased at price at most $\gamma^{-1}p$ by $\gamma^{-1}\epsilon$, i.e., $\pi_i(p+\epsilon) = \pi_i(p) + \gamma^{-1}\epsilon x_i(\lceil\gamma^{-1}(p+\epsilon)/\epsilon\rceil\epsilon)$; (iii) compute demands

[1] notice we are not writing as the area above $p \mapsto x_2(v_1, p)$ as in Myerson-type analysis, but as an area above $p \mapsto \psi(p) := x_2(p, p-)$ as it is usually done for ascending auctions.

as if B were a hard budget: $d_i = \frac{1}{p+\epsilon}[B_i - \pi_i(p+\epsilon)]$ if $p < v_i$ and zero otherwise; (iv) Compute the amount that each player is able to *clinch*, i.e., what is the unallocated amount can be safely given to this player without violating the demand of the other player $\delta_i = \max(0, 1 - x_1(p) - x_2(p) - d_{-i})$; (v) allocate to player i his clinched amount and update his payment: $x_i(p+\epsilon) = x_i(p) + \delta_i$, $\pi_i(p+\epsilon) = \pi_i(p) + p \cdot \delta_i$.

This is not the complete picture yet: we need to describe what happens when demands become negative, since unlike a traditional clinching auction, the prices of previously allocated goods also increase. For the sake of building intuition, let's ignore this issue for a second and assume demands never become negative in this design. Then let's look at the allocation of the auction as $\epsilon \to 0$.

Assume some player already clinched a positive amount. Then, by the definition of clinching, it must be the case that just before the price clock ascended to $p + \epsilon$, the unallocated amount of the good corresponded to the demand of each player: $1 - x_1(p) - x_2(p) = \frac{1}{p}[B - \pi_i(p)]$. Since the setting is symmetric, we refer to $\chi(p) := x_1(p) = x_2(p)$. Now, at price $p + \epsilon$, the amount that each player is able to clinch corresponds to the demand reduction of the other player: the demand is reduced for two reasons: (i) the payment $\pi_i(p)$ increases since the player is forced to pay more for the goods acquired at price $\gamma^{-1}p$; (ii) the price itself increases. So, we can bound δ_i as follows (notice that below we refer to $\gamma^{-1}x(\gamma^{-1}p)$ instead of $x_i(\lceil \gamma^{-1}(p+\epsilon)/\epsilon \rceil \epsilon)$ for simplicity, since we are later taking the limit anyway):

$$\begin{aligned}
\delta_i &= 1 - x_1(p) - x_2(p) - \frac{1}{p+\epsilon}[B - \pi_i(p+\epsilon)] \\
&= \frac{1}{p}[B - \pi_i(p)] - \frac{1}{p+\epsilon}[B - (\pi_i(p) + \gamma^{-1}\epsilon\gamma^{-1}x(\gamma^{-1}p))] \\
&= \epsilon\frac{1}{p+\epsilon}\gamma^{-1}x(\gamma^{-1}p) + \left(\frac{1}{p} - \frac{1}{p+\epsilon}\right) \cdot [B - \pi_i(p)]
\end{aligned}$$

Substituting $B - \pi_i(p) = p(1 - x_1(p) - x_2(p)) = p(1 - 2\chi(p))$ and taking the limit as $\epsilon \to 0$, we get:

$$\chi'(p) = \lim_{\epsilon \to 0} \frac{\delta_i}{\epsilon} = \frac{1}{p}\gamma^{-1}\chi(\gamma^{-1}p) + \frac{1}{p^2} \cdot p(1 - 2\chi(p)) = \frac{1}{p}[1 - 2 \cdot \chi(p) + \gamma^{-1}\chi(\gamma^{-1}p)]$$

which recovers the differential equation in Section 3.

Ascending Auction Description The last observation shows that the auction obtained from the differential equation can at least in some part of the space of valuations be cast as the outcome of an ascending price clock auction. Now, we are ready to make this intuition formal by completing the missing details and showing how to cast the entire auction described as an ascending price clock auction, which we call the *Taxed Ascending Auction*. The auction that we will design will be for slightly more general setting: we will B_i to be different but we will assume the same interest rate γ.

We consider a setting with 2 agents each of them defined by two public parameters B_i and γ and a private value v_i. We will assume in the description that both values v_i are multiples of a small quantity ϵ. Our goal is to design an ascending price clock auction to sell one unit of a divisible good.

The price clock will be represented by a pair (p_1, p_2) which will represent the prices for each of the agents. The clock will start at $(0, 0)$ and increment each price by ϵ in a round-robin fashion, i.e., $(0, 0), (\epsilon, 0), (\epsilon, \epsilon), (2\epsilon, \epsilon), (2\epsilon, 2\epsilon), (3\epsilon, 2\epsilon), \ldots$.

We will keep for each player i a vector $\mathtt{dx}_i[p]$ indexed by price p which indicates how many units he acquired at price p and $\mathtt{pr}_i[p]$ as the price per unit he was charged for those units. In each stage, we will refer to $x_i = \sum_p \mathtt{dx}_i[p]$ and $\pi_i = \sum_p \mathtt{pr}_i[p] \cdot \mathtt{dx}_i[p]$. We will also keep a variable x_*, which we will call the $*$-pool and use to store the goods that were allocated to an agent but had to be de-allocated because the agent could no longer afford them. The $*$-pool is a pool of goods that will be reserved to the higher valued agent (the $*$-player). The identity of the $*$-player will be determined once one of the agents drops out of the auction.

Given $i \in \{1,2\}$ we will denote the other player by $-i$, so if $i=1$, x_i and π_i will denote x_1 and π_1 and x_{-i}, π_{-i} will denote x_2 and π_2. Finally, we will also use the notation $[z]^+$ to denote $\max(z,0)$.

Taxed Ascending Auction

Initialize $\mathtt{dx}_i[p]=0$ and $\mathtt{pr}_i[p]=p$ for all prices p, $p_1=p_2=0$ and $i=1$.

Main Loop: Repeat until $p_1 \le v_1$ or $p_2 \le v_2$.

1. Choose agent in round-robin schedule and increase price: $i = 3-i$, $p_i = p_i + \epsilon$.
2. **Update price for previously allocated goods:** if $p_{-i} < v_{-i}$, then for all prices $p' \le \gamma^{-1} p_i$, update $\mathtt{pr}_i[p'] = \gamma^{-1} p_i$.
3. **Recollections:**
    - if $\gamma^{-1} p > v_i$, collect back all the goods allocated to player i and add them to the pool. Formally: $x_* = x_* + \sum_q x_i[q]$ and $x_i[q]=0$ for all q.
    - if i total payment is larger then B_i, collect back the most expensive goods and add them to the $*$-pool. Formally: if $\pi_i > B_i$, find p' such that $\sum_{q<p'} \mathtt{pr}_i[q] \cdot \mathtt{dx}_i[q] \le B_i$ and $\sum_{q \le p'} \mathtt{pr}_i[q] \cdot \mathtt{dx}_i[q] > B_i$. Let $K_i = \frac{1}{\mathtt{pr}_i[p']}[B_i - \sum_{q<p'} \mathtt{pr}_i[q] \cdot \mathtt{dx}_i[q]]$ be the amount of goods at price p' the player can keep. And update: $x_* = x_* + (x_i[p'] - K_i) + \sum_{q>p'} \mathtt{dx}_i[q]$, $\mathtt{dx}_i[q]=0$ for $q>p'$ and $\mathtt{dx}_i[p'] = K_i$.
4. **Clinching:** The demand of i decreases to $d_i = \frac{1}{p_i}[B_i - \pi_i]$ if $p_i \le v_i$ and $d_i = 0$ otherwise; and compute the clinched amount for player $-i$ as $\delta_{-i} = [1 - x_1 - x_2 - x_* - d_i]^+$ and allocate: $\mathtt{dx}_{-i}[p_{-i}] = \mathtt{dx}_{-i}[p_{-i}] + \delta_{-i}$.
5. **Pool Re-assignment:** if $p_{-i} > v_{-i}$, then $\mathtt{dx}_i[p_i] = \mathtt{dx}_i[p_i] + x_*$, $x_* = 0$.

First we show that the Taxed Ascending Auction has the desired properties:

Theorem 4.1. *The Taxed Ascending Auction is an incentive-compatible, individually rational and Pareto efficient auction for 2 players with taxation functions $\tau_i(\pi_i) = \pi_i$ for $\pi_i \le B_i$ and $\tau_i(\pi_i) = B_i + \gamma^{-1}(\pi_i - B_i)$ for $\pi_i \ge B_i$.*

The first part of the theorem is easy, as it is usually the case in the analysis of ascending auctions. The auction is individually rational because no item is ever given to an agent at a price-per-unit larger then his value. Also, if we ever raise the price of previously allocated goods to a price higher then the agent's valuation, we collect back those goods in step (3).

For incentive compatibility, notice that the only point in which the valuation affects the auction is when the agent's demand drops to zero in step (4). Besides step (4), the agent valuation is used nowhere else. By increasing his value, the agent can only potentially be allocated goods at a higher price per unit then his valuation. Declaring a smaller value can only prevent him from acquiring goods he wants. Also notice that the price increase for previously allocated goods in step (2) and the recollection in step (3) are unaffected by the value of the agent. So no matter which value agent i declares, the price per unit paid in the end will be at least $\gamma^{-1} v_{-i}$. Also note that the re-assignment to the $*$-pool are also not affected by the value declaration of the agents.

Finally, we need to show that the outcome is Pareto-efficient. First we argue that all the good is allocated in the end, i.e., by the end of the auctions $x_1 + x_2 = 1$. The proof of that fact is similar to the one for the traditional clinching auction. First we consider the following invariant:

Lemma 4.2. *In the beginning of each iteration of the main loop, the following invariant holds:* $min(1, \frac{1}{p_i}[B_i - \pi_i]) = 1 - x_1 - x_2 - x_*$.

Proof. This is clearly true for $p = (0, 0)$. Now, we show that this is preserved as an invariant. Notice that in steps (1) and (2), the value of p_i and π_i. Step (3) prevents π_i from being above B_i, so the quantity $\min(1, \frac{1}{p_i}[B_i - \pi_i])$ is non-increasing. If it stays the same, then no clinching happens and since nothing changes for player $-i$, then the invariant continues to hold. If on the other hand, this value decreases, the notice that since $1 - x_1 - x_2 - x_*$ was equal to $\min(1, \frac{1}{p_i}[B_i - \pi_i])$ in the beginning of that iteration of the main loop, then the clinched amount δ_{-i} is equal to the decrease in $\min(1, \frac{1}{p_i}[B_i - \pi_i])$, therefore preserving the invariant. ■

Lemma 4.3. *All goods are allocated in the end of the auction.*

Proof. The previous lemma states that while there are unallocated goods, both players have demand that equals the total amount of unallocated goods. At the first time that one player reduces his demand to zero, the other player still demands the entire amount of unallocated goods, so he will clinch the remainder.

We note that the collection and reassignment of goods, via the $*$-pool doesn't influence this argument, since re-collected goods are eventually re-assigned to the highest valued player. ■

The following lemma, whose proof can be found in Appendix A, completes the proof of Pareto-efficiency of the Taxed Ascending Auction.

Lemma 4.4. *The outcome of the Taxed Ascending Auction satisfies condition (ii) in Lemma 2.3.*

Theorem 4.5. *For the case in which $B_1 = B_2 = B$, the limit of the Taxed Ascending Auction as $\epsilon \to 0$ is the Taxed Differential Auction.*

We postpone a formal proof of Theorem 4.5 to the full version of the paper. We present here a brief intuition of why the equivalence holds. For valuation profiles (v_1, v_2) in regions A and A' of Figure 3, this is straightforward, as both auctions allocate the entire unit to the high value player. For regions B and B' (as well as C and C' below the dashed line) we can use the same argument as in the *Intuition* paragraph in the beginning of this section to show that in the limit as $\epsilon \to 0$ the clinching procedure boils down to the differential equation defining χ.

The remaining two regions C and C' above the dashed line corresponds to the cases in which the budget of the lower valued player reaches B. In the differential version of the auction, this corresponds to $v_2 > \theta(v_1)$ in region C and in the ascending auctions version, it corresponds to units previously allocated to C being assigned to the $*$-pool. Region D corresponds in the ascending auction to the case in which all units assigned to agent 2 have their price increased to $\gamma^{-1} v_1$ such that agent 2 can only keep $B/(\gamma^{-1} v_1)$ units, which is exactly his allocation in region D.

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A Missing Proofs

Proof of Lemma 4.4 : We consider four cases:

1. $v_2 < \gamma^{-1} v_1$. In this case, the price of all items allocated to player 2 are raised to at least $\gamma^{-1} v_1$ in step (2) of the auction, which causes them to be collected in step (1). Therefore the allocation is $(1, 0)$. For this allocation, condition (ii) in Lemma 2.3 states that $v_2 \leq \tau'(\pi_1 -) v_1$ which holds since $v_2 \leq \gamma^{-1} v_1 \leq \tau'(\pi_1 -) v_1$.

2. $v_1 < \gamma^{-1} v_2$. Analogous to the previous item.

3. $\gamma^{-1} v_1 < v_2 \leq v_1$. Consider the first price in which the demand of one of the agents drops to zero. This can happen for one of two reasons:

   (a) price p_2 reaches v_2, his demand drops to zero and player 1 clinches less then his entire demand. The only way this can happen is (by Lemma 4.2) if the demand of 1 was larger then 1, which implies that player 2 didn't have an opportunity to clinch any amount. So the payments are $\pi_1 < B_1$ and $\pi_2 = 0$, implying a Pareto-optimal outcome.

   (b) price p_2 reaches v_2, his demand drops to zero and player 1 clinches his entire demand, spending B_1. In this case, the payments at that moment of the auction are B_1 for player 1 and some $\pi_2 < B_2$ for player 2. The price clock will continue to ascend all the way up to v_1 and no more clinching will happen, but the price of previously allocated units to player 2 keep rising. Either:

      - the price increase won't cause player 2 to pay B_2 or more. In such case the final payments are B_1 for player 1 and $0 < \pi_2 < B_2$. In such case we need to check two inequalities for property (ii) of Lemma 2.3: $v_1 \tau'(\pi_1 -) = v_1 \geq v_2 \tau'(\pi_2 +) = v_2$ and $v_2 \tau'(\pi_2 -) = v_2 \geq v_1 \tau'(pi_1 +) = \gamma^{-1} v_1$, both of which hold by the assumption in this case.
      - the price increase causes player 2 to pay more then B_2 and some units are collected in step (2), assigned to the $*$-pool and then re-assigned later for some price $p \geq v_2$ to player 1, which is the highest value player. By the recollection procedure, the payment of player 2 is exactly B_2 while player 1 has some payment strictly greater B_1 (strictly greater because he is assigned units from the $*$-pool. Now, checking conditions (ii), we get: $v_1 \tau'(\pi_1 -) = \gamma^{-1} v_1 \geq v_2 \tau'(\pi_2 +) = \gamma^{-1} v_2$ and $v_2 \tau'(\pi_2 -) = v_2 \geq v_1 \tau'(pi_1 +) = \gamma^{-1} v_1$, both of which hold by the assumptions in this case.

(c) before reaching v_2, the demand of one player reaches zero because his payment reaches B_i by the increase in price of previously allocated goods that happens in step (2) of the auction. If this happens, immediately the payment of the other agent also reaches B_i in the clinching step. As the clock ascends, both agents potentially have the price of their allocated goods increased and have items collected to the $*$-pool, but the total payment remains B_i. Only the higher valued player (in this case player 1) receives the goods from $*$-pool, so we end up with an allocation in which player 2 pays B_2 and player 1 pays either B_1 or some amount larger or equal to B_1. Condition (ii) in Lemma 2.3 holds by the same argument as in the second part of item (b).

4. $\gamma^{-1} v_2 < v_1 < v_2$ is analogous.

■